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Left-covariant differential calculi on $SL_q(2)$ and $SL_q(3)$

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Abstract

We study (N^2-1)-dimensional left-covariant differential calculi on the quantum group $SL_q(N)$ for which the generators of the quantum Lie algebras annihilate the quantum trace. In this way we obtain one distinguished calculus on $SL_q(2)$ (which corresponds to Woronowicz' 3D-calculus on $SU_q(2)$) and two distinguished calculi on $SL_q(3)$ such that the higher-order calculi give the ordinary differential calculus on $SL(2)$ and $SL(3)$, respectively, in the limit $q\to 1$. Two new differential calculi on $SL_q(3)$ are introduced and developed in detail.

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0. Introduction

After the seminal work of Woronowicz [18], bicovariant differential calculi on quantum groups (Hopf algebras) have been extensively studied in the literature. There is a well developed general theory of such calculi. Bicovariant differential calculi on the quantum group $SL_q(N)$, $N\geq 3$, have been recently classified in [12]. The case of $SL_q(2)$ has been treated before in [15,11]. All calculi occurring in this classification have dimension N^2, i.e. their dimension does not coincide with the dimension (N^2-1) of the corresponding classical Lie group. On the other hand, the first example of a non-commutative differential calculus on a quantum group was Woronowicz' 3D-calculus on $SU_q(2)$ [17]. This is a three-dimensional left-covariant calculus which is not bicovariant. The 3D-calculus is algebraically much

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simpler and in many respects nearer to the classical differential calculus on $SU_q(2)$ than the four-dimensional bicovariant calculi on $SU_q(2)$. This motivates to look for (N^2-1)-dimensional left-covariant differential calculi on $SL_q(N)$. The purpose of this paper is to study the cases $N=2$ and $N=3$ in detail. The main aim of our approach is to follow the classical situation as close as possible.

Let us briefly explain the basic idea of the approach given in this paper. As in [12], we assume that the differentials $\mathrm{d}u^i_j$ of the matrix entries u^i_j generate the left module of 1-forms. Hence the differential d can be expressed as $\mathrm{d}x = \sum(X_{ij} * x)\omega(u^i_j)$ for $x \in SL_q(N)$, where $\omega(u^i_j) := \sum_n \kappa(u^i_n)\,\mathrm{d}u^n_j$ are the left-invariant Maurer–Cartan forms and X_{ij} are linear functionals on $SL_q(N)$ such that $X_{ij}(1) = 0$. In our approach for $N = 2, 3$ the functionals X_{ij} will be chosen from the quantized universal enveloping algebra $\mathcal{U}_q(sl_N)$. For the functionals X_{ij} with $i \neq j$ we take quantum analogues of the corresponding root vectors of sl_N multiplied by some polynomials in the diagonal generators of $\mathcal{U}_q(sl_N)$. We assume that the vector space of left-invariant 1-forms has dimension (N^2-1) and that $X_{ij}(u^r_s) = \delta_{ir}\delta_{js}$ for $i \neq j$. In case of the ordinary differential calculus on $SL(N)$ we have $\sum_i \omega_{ii} = 0$, so it seems to be natural to suppose that $\sum q^{-2i}\omega_{ii} = 0$ in the quantum case. Then all functionals of the corresponding quantum Lie algebra annihilate the quantum trace $U := \sum q^{-2i} u^i_i$. Note that Woronowicz' 3D-calculus on $SU_q(2)$ fits into this scheme, see Section 2 for details.

The paper is organized as follows. Section 1 contains some general results about left-covariant differential calculi on quantum groups which will be needed later. In particular, we describe the construction of the universal higher-order differential calculus associated with a given left-covariant first-order calculus on a quantum group.

In Section 2 we develop four left-covariant differential calculi $(\Gamma_r, \mathrm{d}), r = 1, 2, 3, 4$, on the quantum group $SL_q(2)$ which satisfy the above requirements. All four first-order calculi and quantum Lie algebras give the ordinary differential calculus on $SL(2)$ and the Lie algebra sl_2 when $q \to 1$. However, this changes if we look at the associated higher-order calculi. For only one of these calculi the higher-order calculus yields the classical calculus on $SL(2)$ in the limit $q \to 1$. As might be expected, this is Woronowicz' 3D-calculus on $SU_q(2)$ or more precisely its analogue for $SL_q(2)$. For the other three calculi the 2-form $\omega_2 \wedge \omega_0$ is zero and all 3-forms vanish.

In Sections 3 and 4 we are concerned with left-covariant differential calculi on the quantum group $SL_q(3)$. The functionals X_n and X_{ij} with $|i-j| = 1$ are defined completely similar to the corresponding formulas for the 3D-calculus in Section 2. For the functionals X_{13} and X_{31} we use the Ansatz $X_{13} = X_{12}X_{23} - \alpha X_{23}X_{12}$ and $X_{31} = X_{32}X_{21} - \beta X_{21}X_{23}$ with α and β complex. For arbitrary complex parameters α and β, we obtain a first-order differential calculus on $SL_q(3)$ which fits into the above scheme. It turns out that if $(\alpha, \beta) \neq (q^{-1}, q)$ and $(\alpha, \beta) \neq (q, q^{-1})$, then the 2-form $\omega_{31} \wedge \omega_{13}$ vanishes and hence the space of 2-forms does not yield the corresponding space for the classical differential calculus on $SL(3)$ when $q \to 1$. The differential calculi obtained in the two remaining cases $(\alpha, \beta) = (q, q^{-1})$ and $(\alpha, \beta) = (q^{-1}, q)$ for the parameters α and β are studied in Section 4. In both cases the higher-order calculi give the ordinary higher-order calculus on $SL(3)$ in the limit $q \to 1$. The corresponding formulas show that these two calculi are very close to the classical differential calculi on $SL(3)$ in many respects.

In Section 5 we generalize the two differential calculi on $SL_q(3)$ from Section 4 to $SL_q(N)$. We define two (N^2-1)-dimensional left-covariant first-order differential calculi (Γ_r, d), $r = 1, 2$, over $SL_q(N)$ for which all quantum Lie algebra generators annihilate the quantum trace. Both first-order calculi give the ordinary first-order calculus on $SL(N)$ when $q \to 1$. If $N \geq 4$, this is no longer true for the higher-order calculi.

Throughout this paper q is a non-zero complex number such that $q^2 \neq 1$ and we abbreviate $\lambda := q - q^{-1}$ and $\lambda_+ := q + q^{-1}$. We recall the definition of the quantized universal enveloping algebra $\mathcal{U}_q(sl_N)$, see [4,7]. We shall need it only for $N = 2$ and $N = 3$. The algebra $\mathcal{U}_q(sl_N)$ has $4(N-1)$-generators $k_i, k_i^{-1}, e_i, f_i, i = 1, \ldots, N-1$, with defining relations:

$$
\begin{aligned}
&k_i k_i^{-1} = k^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i e_i = q e_i k_i, \quad k_i f_i = q^{-1} f_i k_i,\\
&e_i f_j - f_j e_i = \delta_{ij} \lambda^{-1} (k_i^2 - k_i^{-2}),\\
&k_i e_j = q^{-1/2} e_j k_i \quad \text{and} \quad k_i f_j = q^{1/2} f_j k_i \qquad \text{if } |i-j| = 1,\\
&e_i^2 e_j - \lambda_+ e_i e_j e_i + e_j e_i^2 = f_i^2 f_j - \lambda_+ f_i f_j f_i + f_j f_i^2 = 0 \quad \text{if } |i-j| = 1,\\
&k_i e_j = e_j k_i, \quad k_i f_j = f_j k_i, \quad e_i e_j = e_j e_i \quad \text{and} \quad f_i f_j = f_j f_i \quad \text{if } |i-j| \geq 2.
\end{aligned}
$$

The Hopf algebra structure of $\mathcal{U}_q(sl_N)$ is given by the comultiplication Δ with $\Delta(k_i) = k_i \otimes k_i$, $\Delta(e_i) = k_i \otimes e_i + e_i \otimes k_i^{-1}$, $\Delta(f_i) = k_i \otimes f_i + f_i \otimes k_i^{-1}$ and the counit ε with $\varepsilon(k_i) = 1$, $\varepsilon(e_i) = \varepsilon(f_i) = 0$. There is a pairing between the Hopf algebras $\mathcal{U}_q(sl_N)$ and $SL_q(N)$ such that $(k_i, u_m^n) = \delta_{nm}$ if $n \neq m$ or $n = m \neq i, i+1$, $(k_i, u_i^i) = q^{1/2}$, $(k_i, u_{i+1}^{i+1}) = q^{-1/2}$, $(e_i, u_m^n) = \delta_{ni}\delta_{m,i+1}$ and $(f_i, u_m^n) = \delta_{n,i+1}\delta_{mi}$, where u_m^n are the matrix entries of the fundamental matrix of $SL_q(N)$.

1. Left-covariant differential calculi on quantum groups

Our basic reference concerning differential calculi on quantum groups is [18]. Let $\mathcal{A}$ be a fixed Hopf algebra with comultiplication Δ, counit ε, antipode κ and unit element 1. Sometimes we use Sweedler's notation $\Delta^{(n)}(a) = a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)}$. For $a \in \mathcal{A}$ we put $\tilde{a} := a - \varepsilon(a)1$.

A *first-order differential calculus* (FODC) over $\mathcal{A}$ is a pair (Γ, d) of an $\mathcal{A}$-bimodule Γ and a linear mapping $\mathrm{d} : \mathcal{A} \to \Gamma$ such that $\mathrm{d}(ab) = \mathrm{d}a \cdot b + a \cdot \mathrm{d}b$ for $a, b \in \mathcal{A}$ and $\Gamma = \mathrm{Lin}\,\{a \cdot \mathrm{d}b \colon a, b \in \mathcal{A}\}$. A FODC (Γ, d) is called *left covariant* if there is a linear mapping $\Delta_{\mathrm{L}} \colon \Gamma \to \mathcal{A} \otimes \Gamma$ for which $\Delta_{\mathrm{L}}(a\,\mathrm{d}b) = \Delta(a)(\mathrm{id} \otimes \mathrm{d})\Delta(b)$, $a, b \in \mathcal{A}$.

Suppose that (Γ, d) is a left-covariant FODC over $\mathcal{A}$. Recall that the canonical projection of Γ into $\Gamma_{\mathrm{inv}} := \{\omega \in \Gamma \colon \Delta_{\mathrm{L}}(\omega) = 1 \otimes \omega\}$ is defined by $P_{\mathrm{inv}}(\mathrm{d}a) = \kappa(a_{(1)})\,\mathrm{d}a_{(2)}$, cf. [18]. We abbreviate $\omega(a) = P_{\mathrm{inv}}(\mathrm{d}a)$. Then $\mathcal{R} := \{x \in \ker \varepsilon \colon \omega(x) = 0\}$ is the *right ideal* of $\ker \varepsilon$ *associated with* (Γ, d). The vector space $\mathcal{X} := \{X \in \mathcal{A}' \colon X(1) = 0$ and $X(x) = 0$ for $x \in \mathcal{R}\}$ is called the *quantum Lie algebra* of the FODC (Γ, d).

Lemma 1. *A vector space $\mathcal{X}$ of linear functionals on $\mathcal{A}$ is the quantum Lie algebra of a left-covariant FODC (Γ, d) if and only if $X(1) = 0$ and $\Delta X - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}'$ for all $X \in \mathcal{X}$.*

Proof. The necessity of the condition $\Delta X - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}'$ follows at once from formula (5.20) in [18]. To prove the sufficiency part, let us note that the above conditions imply that $\mathcal{R} := \{x \in \ker \varepsilon : X(x) = 0 \text{ for } X \in \mathcal{X}\}$ is a right ideal of $\ker \varepsilon$. From the general theory (cf. Theorem 1.5 in [18]) we conclude easily that $\mathcal{R}$ is the right ideal associated with some left-covariant FODC over $\mathcal{A}$. □

The calculus (Γ, d) is uniquely determined by $\mathcal{X}$ (because $\mathcal{R}$ is so) and can be described as follows. Let $\{X_i : i \in I\}$ be a basis of the vector space $\mathcal{X}$ and $\{x_i : i \in I\}$ a set of elements of $\mathcal{A}$ such that $X_i(x_j) = \delta_{ij}$. Then, letting $\omega_i = \omega(x_i)$, we have

$$\mathrm{d}a = \sum (X_i * a)\, \omega_i, \qquad a \in \mathcal{A}.$$

For notational simplicity we shall write $\eta \otimes \zeta$ instead of $\eta \otimes_{\mathcal{A}} \zeta$, where $\eta, \zeta \in \Gamma$. We set for $x \in \mathcal{R}$

$$\mathcal{S}(x) := \sum_{i,j} \left(X_i X_j\right)(x)\, \omega_i \otimes \omega_j .$$

Some properties of the mapping $\mathcal{S} : \mathcal{R} \to \Gamma \otimes_{\mathcal{A}} \Gamma$ are collected in the following:

Lemma 2. *Let P_{inv} denote the canonical projection of the left-covariant bimodule $\Gamma \otimes_{\mathcal{A}} \Gamma$ into $(\Gamma \otimes_{\mathcal{A}} \Gamma)_{\mathrm{inv}}$. For $x \in \mathcal{R}$ and $a \in \mathcal{A}$, we have:*

(i) $\mathcal{S}(x)a = a_{(1)}\mathcal{S}(xa_{(2)})$;

(ii) $\mathcal{S}(xa) = \kappa(a_{(1)})\mathcal{S}(x)a_{(2)} = P_{\mathrm{inv}}(\mathcal{S}(x)a)$.

Proof. We show the first equality of (ii). Recall that $\Delta X_i = \varepsilon \otimes X_i + X_k \otimes f_i^k$, $i \in I$, by formula (5.20) in [18], where f_i^k are functionals on $\mathcal{A}$ such that $\omega_k a = (f_i^k * a)\omega_i$, $a \in \mathcal{A}$. Hence we get

$$\begin{aligned}\mathcal{S}(xa) &= \sum_{i,j} (X_i X_j)(xa)\, \omega_i \otimes \omega_j = \sum_{i,j} (\Delta X_i)(\Delta X_j)(x \otimes a)\, \omega_i \otimes \omega_j \\ &= \sum_{i,j,k,l} (X_k X_l)(x) f_i^k(a_{(1)}) f_j^l(a_{(2)})\, \omega_i \otimes \omega_j .\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\kappa(a_{(1)})\mathcal{S}(x)a_{(2)} &= \sum_{k,l} \kappa(a_{(1)})(X_k X_l)(x)\omega_k \otimes \omega_l a_{(2)} \\ &= \sum_{k,l} \kappa(a_{(1)})(X_k X_l)(x) f_i^k * (f_j^l * a_{(2)})\omega_i \otimes \omega_j \\ &= \sum_{i,j,k,l} \kappa(a_{(1)})a_{(2)} f_i^k(a_{(3)}) f_j^l(a_{(4)})(X_k X_l)(x)\omega_i \otimes \omega_j .\end{aligned}$$

By the Hopf algebra axioms both expressions are equal. The preceding calculation yields $a_{(1)}\mathcal{S}(xa_{(2)}) = a_{(1)}\kappa(a_{(2)})\mathcal{S}(x)a_{(3)} = \mathcal{S}(x)a$ which proves (i). Applying P_{inv} to (i) and using the fact that $P_{\text{inv}}(a\eta) = \varepsilon(a)P_{\text{inv}}(\eta)$ we get the second equality of (ii). □

A *differential calculus* over $\mathcal{A}$ is a pair $(\Gamma^\wedge, \mathrm{d})$ of a graded algebra $\Gamma^\wedge = \bigoplus_{n=0}^{\infty}\Gamma_n^\wedge$ with product $\wedge : \Gamma_n^\wedge \times \Gamma_m^\wedge \to \Gamma_{n+m}^\wedge$ and a linear mapping $\mathrm{d} : \Gamma^\wedge \to \Gamma^\wedge$ of degree one such that $\mathrm{d}^2 = 0$, $\mathrm{d} : \Gamma_n^\wedge \to \Gamma_{n+1}^\wedge$, $\mathrm{d}(\eta \wedge \zeta) = \mathrm{d}\eta \wedge \zeta + (-1)^n \eta \wedge \mathrm{d}\zeta$ for $\eta \in \Gamma_n^\wedge$, $\zeta \in \Gamma_m^\wedge$, $\Gamma_0^\wedge = \mathcal{A}$ and $\Gamma_n^\wedge = \mathrm{Lin}\{a\,\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n : a, a_1, \ldots, a_n \in \mathcal{A}\}$ for $n \in \mathbb{N}$. The definition of left covariance of a differential calculus is similar to the case of first-order calculi. Sometimes we simply write $\eta\zeta$ for $\eta \wedge \zeta$.

For each left-covariant FODC (Γ, d_1) over $\mathcal{A}$ there exists a unique (up to isomorphism) universal left-covariant differential calculus $(\Gamma^\wedge, \mathrm{d})$ over $\mathcal{A}$ such that $\Gamma_1^\wedge = \Gamma$ and $\mathrm{d}\lceil\mathcal{A} = \mathrm{d}_1$. We briefly describe the construction of $(\Gamma^\wedge, \mathrm{d})$.

Let $\Omega = \bigoplus_{n=0}^{\infty}\Omega^n$ be the universal differential envelope of the algebra $\mathcal{A}$, see e.g. [2] or [3]. We have $\Omega^0 = \mathcal{A}$ and $\Omega^n = \mathcal{A} \otimes (\ker\varepsilon)^{\otimes n}$ by identifying $a \otimes a_1 \otimes \cdots \otimes a_n$ and $a\,\mathrm{d}a_1 \ldots \mathrm{d}a_n$. The differential d of Ω is given by $\mathrm{d}(a\,\mathrm{d}a_1 \ldots \mathrm{d}a_n) = \mathrm{d}a\,\mathrm{d}a_1 \ldots \mathrm{d}a_n$. Clearly, (Ω, d) is a differential calculus over the Hopf algebra $\mathcal{A}$. Let $\mathcal{J}(\mathcal{R}) := \Omega\,\omega(\mathcal{R})\Omega + \Omega\,\mathrm{d}\omega(\mathcal{R})\Omega$ be the differential ideal generated by the set $\omega(\mathcal{R})$. Here $\mathcal{R}$ is the right ideal of $\ker\varepsilon$ associated with the given FODC (Γ, d_1). We have $\mathcal{J}(\mathcal{R}) = \sum_n \mathcal{J}_n(\mathcal{R})$, where $\mathcal{J}_n(\mathcal{R}) := \mathcal{J}(\mathcal{R}) \cap \Omega^n$. Obviously, the quotient algebra $\Omega/\mathcal{J}(\mathcal{R}) = \sum_n \Omega^n/\mathcal{J}_n(\mathcal{R})$ endowed with the quotient map of d is also differential calculus over $\mathcal{A}$. By formula (1.23) in [18], $\mathcal{J}_1(\mathcal{R}) = \mathcal{A}\omega(\mathcal{R})$ coincides with the submodule $\mathcal{N}$ occurring in Theorem 1.5 of [18]. Therefore, the first-order calculus $(\Omega_1/\mathcal{J}_1(\mathcal{R}), \mathrm{d})$ of $(\Omega/\mathcal{J}(\mathcal{R}), \mathrm{d})$ is isomorphic to (Γ, d_1). Moreover, it is not difficult to verify that $(\Omega/\mathcal{J}(\mathcal{R}), \mathrm{d})$ is left covariant. From the preceding construction it is clear that $(\Omega/\mathcal{J}(\mathcal{R}), \mathrm{d})$ has the following universal property: If (Ω', d') is another differential calculus over $\mathcal{A}$ such that (Ω_1', d') is isomorphic to the FODC (Γ, d_1), then (Ω', d') is (isomorphic to) a quotient of $(\Omega/\mathcal{J}(\mathcal{R}), \mathrm{d})$ by some differential ideal.

In order to obtain a more explicit description of the calculus $(\Omega/\mathcal{J}, \mathrm{d})$, we shall use a construction of the differential envelope $\Omega = \bigoplus_n \Omega^n$ of the Hopf algebra $\mathcal{A}$ developed in [14]. More details and proofs of all unproven assertions in the following discussion can be found in [14].

Let $\Omega^0 := \mathcal{A}$. For $n \in \mathbb{N}$, we set $\Omega^n := \mathcal{A} \otimes (\ker\varepsilon)^{\otimes n}$ and we write $a\omega(a_1)\ldots\omega(a_n)$ instead of $a \otimes a_1 \otimes \cdots \otimes a_n$. We put $\omega(\lambda 1 + a) = \omega(a)$ for $\lambda \in \mathbb{C}$, $a \in \ker\varepsilon$. The product of $\Omega = \bigoplus_{n=0}^{\infty}\Omega^n$ and the differential d are defined by

$$
\begin{aligned}
&a\omega(a_1)\ldots\omega(a_n)b\,\omega(b_1)\ldots\omega(b_m) \\
&\quad := ab_{(1)}\,\omega(a_1 b_{(2)})\ldots\omega(a_{n-2}b_{(n-1)})\,\omega(a_{n-1}b_{(n)})\,\omega(a_n b_{(n+1)})\,\omega(b_1)\ldots\omega(b_m)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{d}(a\omega(a_1)\ldots\omega(a_n)) := a_{(1)}\omega(a_{(2)})\,\omega(a_1)\ldots\omega(a_n) \\
&\quad + \sum_{i=1}^{n}(-1)^i a\omega(a_1)\ldots\omega(a_{i-1})\omega(a_{i,(1)})\omega(a_{i,(2)})\omega(a_{i+1})\ldots\omega(a_n),
\end{aligned}
$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \ker\varepsilon$ and $a, b \in \mathcal{A}$. (In order to motivate these formulas, we recall that for any left-covariant differential calculus over $\mathcal{A}$ we have $\mathrm{d}a = a_{(1)}\omega(a_{(2)})$, $\mathrm{d}\omega(a) = -\omega(a_{(1)}) \wedge \omega(a_{(2)})$ and $\omega(b)c = c_{(1)}\omega(bc_{(2)})$ for $a, c \in \mathcal{A}$ and $b \in \ker\varepsilon$.) It can be shown that the pair (Ω, d) endowed with the above definitions becomes a left-covariant differential calculus over $\mathcal{A}$ which is (isomorphic to) the differential envelope of $\mathcal{A}$. For $a \in \mathcal{R}$, the element $\omega(a)$ of Ω is equal to $\kappa(a_{(1)})\,\mathrm{d}a_{(2)}$ which justifies to use the notation $\omega(a)$. Obviously, the kernel of the map $\omega : \mathcal{A} \to \Omega$ is $\mathbb{C} \cdot 1$. Let $\mathcal{R}$, $\{X_i\}$, $\{x_i\}$, $\{\omega_i\}$ and $\mathcal{S}$ be as defined above for the left-covariant FODC (Γ, d_1). We put

$$\mathcal{S}_u(x) := \sum_{i,j}(X_iX_j)(x)\omega_i\omega_j \quad \text{for } x \in \mathcal{R},$$

where the product $\omega_i\omega_j$ is taken in the algebra Ω. For $a \in \mathcal{A}$, the element $\tilde{a} - \sum_i X_i(a)\tilde{x}_i$ is annihilated by $\mathcal{X}$ and by ε, so it belongs to $\mathcal{R}$ and hence $\omega(a) - \sum_i X_i(a)\omega_i \in \omega(\mathcal{R})$. Therefore, since $\mathrm{d}\omega(x) = -\omega(x_{(1)})\omega(x_{(2)})$, we obtain

$$\begin{aligned}\mathrm{d}\omega(x) + \mathcal{S}_u(x) &= -\left(\omega(x_{(1)}) - \sum_i X_i(x_{(1)})\omega_i\right)\omega(x_{(2)}) \\ &\quad + \sum_i X_i(x_{(1)})\omega_i\left(\sum_j X_j(x_{(2)})\omega_j - \omega(x_{(2)})\right) \\ &\in \omega(\mathcal{R})\Omega^1 + \Omega^1\omega(\mathcal{R}) \quad \text{for } x \in \mathcal{R}.\end{aligned}$$

Hence the differential ideal $\mathcal{J}(\mathcal{R})$ is generated by the sets $\omega(\mathcal{R})$ and $\mathcal{S}_u(\mathcal{R})$. Next we define the exterior algebra for the FODC (Γ, d_1). Let $\Gamma^{\otimes 0} := \mathcal{A}$, $\Gamma^{\otimes n} := \Gamma \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Gamma$ (n times) for $n \in \mathbb{N}$ and let $\Gamma^{\otimes} := \bigoplus_{n=0}^{\infty}\Gamma^{\otimes n}$ be the tensor algebra of Γ over $\mathcal{A}$. We denote by $S = \bigoplus_{n=2}^{\infty} S_n$ the two-sided ideal of the algebra $\Gamma^{\otimes}$ generated by the set $\mathcal{S}(\mathcal{R})$. The quotient algebra $\Gamma^{\wedge} = \Gamma^{\otimes}/S$ is called the *exterior algebra* over $\mathcal{A}$ *for the* FODC (Γ, d_1). Clearly, $\Gamma^{\wedge}$ is also a graded algebra $\Gamma^{\wedge} = \bigoplus_{n=0}^{\infty}\Gamma_n^{\wedge}$ with $\Gamma_0^{\wedge} = \mathcal{A}$, $\Gamma_1^{\wedge} = \Gamma$ and $\Gamma_n^{\wedge} = \Gamma^{\otimes n}/S_n$ for $n \in \mathbb{N}$. The product of $\Gamma^{\wedge}$ is denoted by $\wedge$.

For $x \in \ker\varepsilon$, let $[x]$ denote the coset $x + \mathcal{R}$. Let π be the product of the mapping $\pi_1 : \Omega \to \Gamma^{\otimes}$ defined by $\pi_1(a) = a$, $\pi_1(a\omega(a_1)\ldots\omega(a_n)) = a\omega([a_1])\ldots\omega([a_n])$ and the quotient map from $\Gamma^{\otimes}$ onto $\Gamma^{\wedge}$. Then π is an algebra homomorphism of Ω onto $\Gamma^{\wedge}$. It can be shown that the kernel of π is the two-sided ideal in Ω generated by $\omega(\mathcal{R})$ and $\mathcal{S}_u(\mathcal{R})$. Therefore, by the paragraph before last, $\ker\pi = \mathcal{J}(\mathcal{R})$. Hence the quotient algebra $\Omega/\mathcal{J}(\mathcal{R})$ and the exterior algebra $\Gamma^{\wedge}$ are isomorphic. We define $\mathrm{d}\pi(a) := \mathrm{d}a$ for $a \in \Gamma$. Then (Γ^1, d) is a left-covariant differential calculus over $\mathcal{A}$ which is isomorphic to the calculus $(\Omega/\mathcal{J}(\mathcal{R}), \mathrm{d})$. By construction, the first-order calculus $(\Gamma_1^{\wedge}, \mathrm{d})$ of $(\Gamma^{\wedge}, \mathrm{d})$ is the given FODC (Γ, d_1). We call $(\Gamma^{\wedge}, \mathrm{d})$ (or likewise $(\Omega/\mathcal{J}(\mathcal{R}), \mathrm{d})$ the *universal differential calculus associated with the* FODC (Γ, d_1).

By definition, $S_2 = \mathcal{A}\mathcal{S}(\mathcal{R})\mathcal{A}$. From Lemma 2 (i), we see that $S_2 = \mathcal{A}\mathcal{S}(\mathcal{R})$ is an $\mathcal{A}$-subbimodule of $\Gamma \otimes_{\mathcal{A}} \Gamma$ and hence a left-covariant bimodule over $\mathcal{A}$. Therefore, each basis (ζ_n) of the vector space $\mathcal{S}(\mathcal{R})$ of symmetric elements is a free left module basis for S_2, i.e. any element ζ of S_2 can be written as $\zeta = \sum_n a_n \zeta_n$ with elements $a_n \in \mathcal{A}$ uniquely determined by ζ.

2. Left-covariant differential calculi on $SL_q(2)$

In this section we denote the matrix entries $u^1_1, u^1_2, u^2_1, u^2_2$ for the quantum group $SL_q(2)$ by a, b, c, d, respectively, and the generators of $\mathcal{U}_q(sl_2)$ by k, k^{-1}, e, f. Our aim is to study left-covariant differential calculi (Γ, d) on $\mathcal{A} := SL_q(2)$ of the form

$$\mathrm{d}x = \sum_{i=0}^{2} (X_i * x)\, \omega_i, \qquad x \in SL_q(2), \tag{2.1}$$

where ω_0, ω_1 and ω_2 are left-invariant 1-forms and X_0, X_1 and X_2 are linear functionals from $\mathcal{U}_q(sl(2))$ satisfying

$$\begin{aligned} &X_1(a) = X_0(b) = X_2(c) = 1,\\ &X_1(b) = X_1(c) = X_0(a) = X_0(c) = X_2(a) = X_2(b) = 0; \end{aligned} \tag{2.2}$$

and

$$X_i(1) = X_i(q^{-2}a + q^{-4}d) = 0 \quad \text{for } i = 0, 1, 2\,. \tag{2.3}$$

From the pairing between $\mathcal{U}_q(sl_2)$ and $SL_q(2)$ it follows that arbitrary linear functionals $X_i \in \mathcal{U}_q(sl(2)), i = 0, 1, 2$, satisfying (2.2) and (2.3) can be written as $X_1 = efp_{11}(k) + p_{12}(k) + X'_1$, $X_0 = ep_0(k) + X'_0$ and $X_2 = fp_2(k) + X'_2$, where p_{11}, p_{12}, p_0, p_2 are Laurent polynomials in k and X'_1, X'_0, X'_2 annihilate all four matrix entries a, b, c, d. This suggests to consider the following Ansatz:

$$X_1 = efp_{11}(k) + p_{12}(k), \qquad X_0 = ep_0(k), \qquad X_2 = fp_2(k) \tag{2.4}$$

with polynomials p_{11}, p_{12}, p_0 and p_2 in k and k^{-1}.

Theorem 1. *There are precisely four non-isomorphic three-dimensional left-covariant differential calculi* $(\Gamma_r, \mathrm{d}), r = 1, 2, 3, 4$, *satisfying* (2.1)–(2.3) *obtained by the Ansatz* (2.4). *The right ideal* $\mathcal{R}_r$ *of* $\ker \varepsilon$ *associated with the calculus* (Γ_r, d) *is generated by six elements*

$$a + q^{-2}d - (1 + q^{-2})1, \quad b^2, \quad c^2, \quad bc, \quad (a - \gamma_{0r})b, \quad (a - \gamma_{2r})c,$$

where γ_{0r} *and* γ_{2r} *are the coefficients given by* $\gamma_{01} = \gamma_{02} = 1, \gamma_{03} = \gamma_{04} = q^{-2}, \gamma_{21} = \gamma_{23} = 1$ *and* $\gamma_{22} = \gamma_{24} = q^{-2}$.

Proof. We first suppose that (Γ, d) is a differential calculus such that (2.1)–(2.4) are fulfilled. Let $\mathcal{R}$ be its associated right ideal of $\ker \varepsilon$. From formulas (2.1)–(2.4) and from the pairing between $\mathcal{U}_q(sl_2)$ and $SL_q(2)$ we compute easily that b^2 and c^2 are in $\mathcal{R}$ and that there are complex numbers $\gamma_1, \gamma_0, \gamma_2$ such that $bc - \gamma_1(a - 1)$, $ab - \gamma_0 b$ and $ac - \gamma_2 c$ are in $\mathcal{R}$. As usual, we shall write $x \equiv y$ if $x - y \in \mathcal{R}$. By (2.3), $a - 1 \equiv -q^{-2}(d - 1)$. Thus

$$
\begin{aligned}
0 &\equiv (bc - \gamma_1(a-1))(a-1) = q^{-2}abc - bc - \gamma_1(a-1)^2 \\
&= \gamma_0 q^{-2}bc - bc - \gamma_1(a-1)^2 = (\gamma_0 q^{-2} - 1)\gamma_1(a-1) - \gamma_1(a-1)^2 \\
&= -\gamma_1(a-1)(a - \gamma_0 q^{-2}1) = \gamma_1 q^{-2}(d-1)(a - \gamma_0 q^{-2}1) \\
&= \gamma_1 q^{-2}(da - a - \gamma_0 q^{-2}(d-1)) = \gamma_1 q^{-2}(q^{-1}bc + 1 - a - \gamma_0 q^{-2}(d-1)) \\
&\equiv \gamma_1 q^{-2}(\gamma_1 q^{-1}(a-1) + 1 - a + \gamma_0(a-1)) = \gamma_1 q^{-2}(\gamma_1 q^{-1} + \gamma_0 - 1)(a-1)
\end{aligned}
$$

and

$$0 \equiv (bc - \gamma_1(a-1))b = b^2c - \gamma_1 ab + \gamma_1 b \equiv \gamma_1(1-\gamma_0)b\,.$$

Since $a-1$, b and c are not in $\mathcal{R}$ by (2.2), we get $\gamma_1(\gamma_1 q^{-1} + \gamma_0 - 1) = 0$ and $\gamma_1(1-\gamma_0) = 0$, so that $\gamma_1 = 0$. This implies that $da - 1 \equiv 0$ and hence $q^2(a-1)(a-q^{-2}) = (d-1)(q^{-2} - a) \equiv -1 + a + q^{-2}(d-1) \equiv 0$, so $0 \equiv (a-1)(a-q^{-2})b = qaba - (1+q^{-2})ab + q^{-2}b = (\gamma_0 - 1)(\gamma_0 - q^{-2})b$ which yields $(\gamma_0-1)(\gamma_0-q^{-2}) = 0$. Similarly we obtain $(\gamma_2 - 1)(\gamma_2 - q^{-2}) = 0$. The two latter equations imply that $\mathcal{R}$ contains one of the right ideals $\mathcal{R}_r$, $r = 1, 2, 3, 4$. From (2.3), ω_0, ω_1 and ω_2 are linearly independent. Hence we have codim $\mathcal{R}$ = dim $\Gamma_{\text{inv}} \geq 3$. Since codim $\mathcal{R}_r \leq 3$ by the definition of $\mathcal{R}_r$, we conclude that $\mathcal{R} = \mathcal{R}_r$.

To complete the proof of Theorem 1, we have to construct the first-order differential calculi (Γ_r, d) having the desired properties. For this let $\mathcal{X}_r$ denote the linear span of functionals $X_1, X_0, X_2 \in \mathcal{U}_q(sl_2)$, where

$$
\begin{array}{llll}
X_0 := q^{-1/2}ek^{-1} & \text{for } r = 1, 2; & X_0 := q^{-5/2}ek^{-5} & \text{for } r = 3, 4; \\
X_2 := q^{1/2}fk^{-1} & \text{for } r = 1, 3; & X_2 := q^{5/2}fk^{-5} & \text{for } r = 2, 4; \\
X_1 := q\lambda^{-1}(\varepsilon - k^{-4}) & \text{for } r = 1, 2, 3, 4\,. & &
\end{array}
$$

From these definitions and from the comultiplication in $\mathcal{U}_q(sl_2)$ we obtain:

$$
\begin{aligned}
&\Delta X_j = \varepsilon \otimes X_j + X_j \otimes k^{-2} \quad \text{for } r = 1, 2, \ \ j = 0 \ \text{ and } \ r = 1, \ \ j = 2, \\
&\Delta X_j = \varepsilon \otimes X_j + X_j \otimes k^{-6} + (q^{-2} - 1)X_1 \otimes X_j \\
&\qquad \text{for } r = 3, 4, \ \ j = 0 \ \text{ and } \ r = 2, 4, \ \ j = 2, \\
&\Delta X_1 = \varepsilon \otimes X_1 + X_1 \otimes k^{-4} \quad \text{for } r = 1, 2, 3, 4\,.
\end{aligned}
$$

This shows that $\Delta X - \varepsilon \otimes X \in \mathcal{X}_r \otimes \mathcal{A}'$ for all $X \in \mathcal{X}_r$. Therefore, by Lemma 1, each vector space $\mathcal{X}_r$ defines a left-covariant first-order differential calculus (Γ_r, d) over $SL_q(2)$. Let $\mathcal{R}_r'$ denote the right ideal of ker ε associated with the calculus (Γ_r, d). One verifies that the functionals X_0, X_1, X_2 for the calculus (Γ_r, d) have the properties (2.2)–(2.3) and that they annihilate the six generators of the right ideal $\mathcal{R}_r$. Hence $\mathcal{R}_r' \subseteq \mathcal{R}_r$. Since codim $\mathcal{R}_r' = \dim(\Gamma_r)_{\text{inv}} = 3$ and codim $\mathcal{R}_r \leq 3$, we have $\mathcal{R}_r' = \mathcal{R}_r$. This completes the proof of Theorem 1. □

We now describe the structure of the four differential calculi (Γ_r, d), $r = 1, 2, 3, 4$, more in detail. By the general theory [18], the above formulas for the comultiplication of X_j lead to the following commutation rules between matrix entries and 1-forms:

$$\omega_j a = q^{-1} a\omega_j, \quad \omega_j b = qb\omega_j, \quad \omega_j c = q^{-1} c\omega_j, \quad \omega_j d = qd\omega_j$$
$$\text{for } r = 1,2, \;\; j = 0 \text{ and } r = 1,3, \;\; j = 2,$$
$$\omega_j a = q^{-3} a\omega_j, \quad \omega_j b = q^3 b\omega_j, \quad \omega_j c = q^{-3} c\omega_j, \quad \omega_j d = q^3 d\omega_j$$
$$\text{for } r = 3,4, \;\; j = 0 \text{ and } r = 2,4, \;\; j = 2,$$
$$\omega_1 a = q^{-2} a\omega_1, \quad \omega_1 c = q^{-2} c\omega_1 \qquad \text{for } r = 1,3,$$
$$\omega_1 b = q^2 b\omega_1, \quad \omega_1 d = q^2 d\omega_1 \qquad \text{for } r = 1,2,$$
$$\omega_1 a = q^{-2} a\omega_1 + (q^{-2} - 1) b\omega_2, \quad \omega_1 c = q^{-2} c\omega_1 + (q^{-2} - 1) d\omega_2 \text{ for } r = 2,4,$$
$$\omega_1 b = q^2 b\omega_1 + (q^{-2} - 1) a\omega_0, \quad \omega_1 d = q^2 d\omega_1 + (q^{-2} - 1) c\omega_0 \quad \text{for } r = 3,4\,.$$

Recall that $\omega(a) = P_{\text{inv}}(\mathrm{d}a) = \kappa(a_{(1)})\,\mathrm{d}a_{(2)}$ if $\Delta(a) = a_{(1)} \otimes a_{(2)}$. From formulas (2.2) we compute that for all four calculi

$$\omega_1 = \omega(a), \qquad \omega_0 = \omega(b), \qquad \omega_2 = \omega(c),$$

and

$$\mathrm{d}a = b\omega_0 + a\omega_1, \qquad \mathrm{d}b = a\omega_2 - q^2 b\omega_1,$$
$$\mathrm{d}c = c\omega_1 + \mathrm{d}\omega_0, \qquad \mathrm{d}d = -q^2 d\omega_1 + c\omega_2.$$

According to the general theory [18], the calculus (Γ_r, d) is a $*$-calculus for an algebra involution $x \to x^*$ on $SL_q(2)$ if and only if $\kappa(x)^* \in \mathcal{R}_r$ for all $x \in \mathcal{R}_r$. Obviously, it suffices to check this condition for the six generators of the right ideal $\mathcal{R}_r$. The four calculi (Γ_r, d), $r = 1,2,3,4$, are $*$-calculi for the Hopf $*$-algebra $SL_q(2,\mathbb{R})$, $|q| = 1$, while only (Γ_1, d) and (Γ_4, d) are $*$-calculi for the real forms $SU_q(2)$ and $SU_q(1,1)$, $q \in \mathbb{R}$, of the quantum group $SL_q(2)$. However, for the Hopf $*$-algebras $SU_q(2)$ and $SU_q(1,1)$ we have $\kappa(x)^* \in \mathcal{R}_2$ for $x \in \mathcal{R}_3$ and $\kappa(x)^* \in \mathcal{R}_3$ for $x \in \mathcal{R}_2$. Moreover, we have $\varphi(\mathcal{R}_2) = \mathcal{R}_3$ and $\varphi(\mathcal{R}_3) = \mathcal{R}_2$, where φ denotes the algebra automorphism of $SL_q(2)$ which fixes a and d and interchanges b and c.

Next we consider the commutation rules between the generators X_0, X_1, X_2 of the quantum Lie algebra $\mathcal{X}_r$ of the calculus (Γ_r, d). We have

$$\begin{aligned}
q^2 X_1 X_0 - q^{-2} X_0 X_1 &= (1+q^2) X_0 \quad \text{for } r = 1,2,3,4,\\
q^2 X_2 X_1 - q^{-2} X_1 X_2 &= (1+q^2) X_2 \quad \text{for } r = 1,2,3,4,\\
q X_2 X_0 - q^{-1} X_0 X_2 &= -q^{-1} X_1 \quad \text{for } r = 1,\\
q^3 X_2 X_0 - q^{-3} X_0 X_2 &= -q^{-1} X_1 + q^{-2} \lambda X_1^2 \quad \text{for } r = 2,3,\\
q^5 X_2 X_0 - q^{-5} X_0 X_2 &= -q^{-1} X_1 + 2q^{-2} \lambda X_1^2 - q^{-3} \lambda^2 X_1^3 \quad \text{for } r = 4.
\end{aligned}$$

What about the classical limits $q \to 1$ of the calculi (Γ_r, d)? If we keep the basis $\{\omega_0, \omega_1, \omega_2\}$ of $(\Gamma_r)_{\text{inv}}$ fixed, all above equations make sense in the limit $q \to 1$ and we obtain the classical first-order differential calculus on the Lie group $SL(2)$. That is, all four calculi (Γ_r, d), $r = 1,2,3,4$, can be considered as deformations of the classical first-order differential calculus on $SL(2)$. In particular, the preceding equations for $r = 1$, $r = 2,3$ and $r = 4$ define three deformations of the Lie algebra sl_2. Note that the quadratic and cubic terms of X_1 in the two last equations vanish in the limit $q \to 1$. It might be worth mentioning that the quantum Lie algebras $\mathcal{X}_2$ and $\mathcal{X}_3$ are isomorphic (because the commutation rules of the generators X_0, X_1, X_2 for $r = 2$ and $r = 3$ are the same), but the right ideals $\mathcal{R}_2$ and $\mathcal{R}_3$ are different and hence the differential calculi (Γ_2, d) and (Γ_3, d) are not ismorphic. Clearly, for the quantum group $SU_q(2)$ the differential calculus (Γ_1, d) is nothing but the 3D-calculus discovered by Woronowicz [17], because the right ideal $\mathcal{R}_1$ coincides with the right ideal of the 3D-calculus, cf. formula (2.27) in [17]. (The slight differences between some of our formulas stated above and the corresponding formulas in [17] stem from the fact that we assumed $X_2(c) = 1$ by (2.2), while $X_2(c) = -q$ by formula (2.3) in [17].) Now we turn to the higher-order differential calculi.

Lemma 3. *If $q^{12} \neq 1$, then $\omega_0 \otimes \omega_2 \in \mathcal{S}(\mathcal{R}_r)$ for $r = 2, 3, 4$.*

Proof. Obviously, $a^2 - (1 + q^{-2})a + q^{-2}1 \in \mathcal{R}_r$. Easy computations yield

$$\mathcal{S}(a^2 - (1+q^{-2})a+q^{-2}1) = (1+q^{-2})(q^{-2}\omega_1 \otimes \omega_1+(\gamma_{0r}\gamma_{2r} - 1)\omega_0 \otimes \omega_2)$$

and

$$\mathcal{S}(a + q^{-2}d - (1 + q^{-2})1) = (1+q^2)\omega_1 \otimes \omega_1 + \omega_0 \otimes \omega_2 + q^{-2}\omega_2 \otimes \omega_0$$

for $r = 2, 3, 4$, so that

$$\begin{aligned} &q^6\omega_0 \otimes \omega_2 + \omega_2 \otimes \omega_0 \in \mathcal{S}(\mathcal{R}_4), \\ &(q^6 + q^4 - 1)\omega_0 \otimes \omega_2+\omega_2 \otimes \omega_0 \in \mathcal{S}(\mathcal{R}_2)\cap\mathcal{S}(\mathcal{R}_3). \end{aligned} \tag{2.5}$$

Let $r = 2$. Then we have $ab - b \in \mathcal{R}_2$ and $\mathcal{S}(ab - b) = \omega_0 \otimes \omega_1 + q^{-2}\omega_1 \otimes \omega_0$. Using Lemma 2(ii), we compute

$$\begin{aligned} \mathcal{S}((ab - b)c) &= P_{\mathrm{inv}}(\mathcal{S}(ab - b)c) = P_{\mathrm{inv}}(\omega_0 \otimes \omega_1 c + q^{-2}\omega_1 \otimes \omega_0 c) \\ &= P_{\mathrm{inv}}(q^{-3}c\omega_0 \otimes \omega_1 + q(q^{-2} - 1)d\omega_0 \otimes \omega_2 \\ &\qquad + q^{-5}c\omega_1 \otimes \omega_0 + q^{-3}(q^{-2} - 1)d\omega_2 \otimes \omega_0) \\ &= (q^{-2} - 1)(q\omega_0 \otimes \omega_2 + q^{-3}\omega_2 \otimes \omega_0) \in \mathcal{S}(\mathcal{R}_2) . \end{aligned} \tag{2.6}$$

Since $q^6 \neq 1$, (2.5) and (2.6) imply that $\omega_0 \otimes \omega_2 \in \mathcal{S}(\mathcal{R}_2)$. Interchanging the role of b and c we get $\omega_0 \otimes \omega_2 \in \mathcal{S}(\mathcal{R}_3)$. For $r = 4$, we have $(ac - q^{-2}c)b \in \mathcal{R}_4$ and

$$\begin{aligned} &\mathcal{S}((ac - q^{-2}c)b) = P_{\mathrm{inv}}((\omega_1 \otimes \omega_2 + q^{-4}\omega_2 \otimes \omega_1)b), \\ &P_{\mathrm{inv}}(q^5b\omega_1 \otimes \omega_2 + q^3(q^{-2} - 1)a\omega_0 \otimes \omega_2 + qb\omega_2 \otimes \omega_1 \\ &\qquad + q^{-7}(q^{-2} - 1)a\omega_2 \otimes \omega_0) = (q^{-2} - 1)(q^3\omega_0 \otimes \omega_2 + q^{-7}\omega_2 \otimes \omega_0) . \end{aligned} \tag{2.7}$$

From (2.5) and (2.7) we obtain $\omega_0 \otimes \omega_2 \in \mathcal{S}(\mathcal{R}_4)$, because we assumed that $q^{12} \neq 1$. □

Therefore, in contrast to the classical case, the 2-form $\omega_0 \wedge \omega_2$ is zero for all three calculi (Γ_r, d), $r = 2, 3, 4$. In particular, this implies that all 3-forms vanish and that the differential of ω_1 is zero. Hence the higher-order calculi of (Γ_r, d), $r = 2, 3, 4$, do *not* give the ordinary differential calculus on $SL(2)$ when $q \to 1$. Recall that the calculus

(Γ_1, d) is 3D-calculus of Woronowicz [17]. As shown in [17], the higher-order calculus of the 3D-calculus yields the ordinary differential calculus on $SU(2)$ (and on $SL(2)$) in the limit $q \to 1$. Thus the considerations in this section emphasize the distinguished role of Woronowicz' 3D-calculus on $SU_q(2)$ resp. of the calculus (Γ_1, d) on $SL_q(2)$ among all three-dimensional left-covariant differential calculi on $SU_q(2)$ resp. $SL_q(2)$.

Remark . The functionals $\tilde{X}_0 := q^{1/2}ek$, $\tilde{X}_2 := q^{-1/2}fk$ and $\tilde{X}_1 := q\lambda^{-1}(\varepsilon - k^4)$ also satisfy the commutation relations of the quantum Lie algebra $\mathcal{X}_1$. This presentation has been found by Sudbery [16], see e.g. [10]. The functionals $\{\tilde{X}_0, \tilde{X}_1, \tilde{X}_2\}$ define another left-covariant FODC $(\tilde{\Gamma}, \mathrm{d})$ on $SL_q(2)$. Since $\tilde{X}_1$ does not annihilate the quantum trace $q^{-2}a + q^{-4}d$, the right ideal of $\ker\varepsilon$ associated with $(\tilde{\Gamma}, \mathrm{d})$ is different from $\mathcal{R}_1$, so that the first-order calculi $(\tilde{\Gamma}, \mathrm{d})$ and (Γ_1, d) are not isomorphic.

3. Left-covariant differential calculi on $SL_q(3)$

In this and the following section we consider left-covariant differential calculi (Γ, d) over $\mathcal{A} = SL_q(3)$ of the form

$$\mathrm{d}x = \sum_{i,j=1;i\neq j}^{3} (X_{ij} * x)\, \omega_{ij} + \sum_{n=1}^{2} (X_n * x)\, \omega_n, \qquad x \in SL_q(3). \tag{3.1}$$

Here ω_{ij} and ω_n are left-invariant 1-forms and X_{ij} and X_n are linear functionals of $\mathcal{U}_q(sl_3)$ such that

$$\begin{aligned} X_{ij}(1) &= 0 \quad \text{and} \quad X_{ij}(u^r_s) = \delta_{ir}\delta_{js} \qquad \text{for } i \neq j, \\ X_n(u^r_s) &= 0 \quad \text{for } r \neq s \quad \text{and} \quad X_n(1) = X_n(U) = 0. \end{aligned} \tag{3.2}$$

Recall that $U := \sum_{i=1}^{3} q^{-2i} u^i_i$ is the quantum trace. We define

$$X_i := q\lambda^{-1}(\varepsilon - k_i^{-4}), \quad X_{i,i+1} := q^{-1/2} e_i k_i^{-1}, \quad X_{i+1,i} := q^{1/2} f_i k_i^{-1}, \quad \text{for } i = 1, 2. \tag{3.3}$$

Let α and β be complex numbers. We set

$$X_{13} = X_{12}X_{23} - \alpha X_{23}X_{12} \quad \text{and} \quad X_{31} = X_{32}X_{21} - \beta X_{21}X_{32}.$$

Then all linear functionals X_{ij} and X_n satisfy conditions (3.2). Let $\mathcal{X}$ denote the vector space generated by these functionals. We compute

$$\begin{aligned} \Delta X_n &= \varepsilon \otimes X_n + X_n \otimes k_n^{-4}, \qquad \Delta X_{ij} = \varepsilon \otimes X_{ij} + X_{ij} \otimes k_j^{-2} \quad \text{for } |i-j| = 1, \\ \Delta X_{13} &= \varepsilon \otimes X_{13} + X_{13} \otimes (k_1k_2)^{-2} + (q-\alpha) X_{12} \otimes X_{23} k_1^{-2} \\ &\quad + (1-\alpha q) X_{23} \otimes X_{12} k_2^{-2}, \end{aligned}$$

and

$$\Delta X_{31} = \varepsilon \otimes X_{31} + X_{31} \otimes (k_1k_2)^{-2} + (1-\beta q^{-1})X_{21} \otimes X_{32}k_1^{-2} + (q^{-1}-\beta)X_{32} \otimes X_{21}k_2^{-2}\,.$$

That is, we have $\Delta X - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}'$ for all $X \in \mathcal{X}$. Therefore, by Lemma 1, the above Ansatz gives a left-covariant differential calculus (Γ, d) over $\mathcal{A} = SL_q(3)$. From the formulas for ΔX_{ij} and ΔX_n we see that the corresponding homomorphism f of the algebra $SL_q(3)$ (as defined in Theorem 2.1, 3., in [18]) decomposes into a direct sum of an upper triangular part, a lower triangular part and a diagonal part. Using the pairing between $\mathcal{U}_q(sl_3)$ and $SL_q(3)$ we obtain the following commutation relations between matrix entries and 1-forms:

$$\begin{aligned}
\omega_{12}u^i_j &= q^{-\delta_{j1}+\delta_{j2}}u^i_j\omega_{12} + \delta_{j3}(q-\alpha)u^i_2\omega_{13},\\
\omega_{23}u^i_j &= q^{-\delta_{j2}+\delta_{j3}}u^i_j\omega_{23} + \delta_{j2}(q^{-1}-\alpha)u^i_1\omega_{13},\\
\omega_{13}u^i_j &= q^{-\delta_{j1}+\delta_{j3}}u^i_j\omega_{13},\\
\omega_{21}u^i_j &= q^{-\delta_{j1}+\delta_{j2}}u^i_j\omega_{21} + \delta_{j2}(q-\beta)u^i_3\omega_{31},\\
\omega_{32}u^i_j &= q^{-\delta_{j2}+\delta_{j3}}u^i_j\omega_{32} + \delta_{j1}(q^{-1}-\beta)u^i_2\omega_{31},\\
\omega_{31}u^i_j &= q^{-\delta_{j1}+\delta_{j3}}u^i_j\omega_{31} \quad \text{and} \quad \omega_n u^i_j = q^{-2\delta_{nj}+2\delta_{n+1,j}}u^i_j\omega_n\,.
\end{aligned}$$

In particular, if $\alpha = \alpha(q)$ and $\beta = \beta(q)$ are functions of q such that their limits are equal to 1 as $q \to 1$, then the calculus (Γ, d) gives the ordinary differential calculus on $SL(2)$ in the limit $q \to 1$.

From the general theory [18] we know that the right ideal $\mathcal{R}$ of $\ker\varepsilon$ associated with a FODC plays a crucial role. For the calculus (Γ, d) defined above the right ideal $\mathcal{R}$ is generated by the following elements:

$$\begin{aligned}
&u^r_s u^i_j \quad \text{for } r \neq s,\ r \neq j,\ i \neq j,\ i \neq s;\\
&u^r_r u^i_j - u^i_j \quad \text{for } i \neq j;\\
&u^i_j u^j_i \quad \text{for } i \neq j; \qquad u^2_1u^1_3; \qquad u^3_1u^1_2; \qquad u^1_3u^3_2; \qquad u^2_3u^3_1;\\
&u^2_3u^1_2 - (q^{-1}-\alpha)u^1_3; \qquad u^2_1u^3_2 - (q-\beta)u^3_1;\\
&u^1_1u^3_3 - u^1_1 - u^3_3 + 1; \qquad u^2_2u^1_1 + q^{-2}u^3_3 - (q^{-2}+1)1;\\
&u^3_3u^2_2 - u^2_2 - q^{-2}u^3_3 + q^{-2}1; \qquad u^1_1u^1_1 - (1+q^{-2})u^1_1 + q^{-2}1;\\
&u^2_2u^2_2 - (1+q^{-2})u^2_2 + (q^4-1)u^1_1 - (q^4-1-q^{-2})1;\\
&u^3_3u^3_3 - (1+q^{-2})u^3_3 + q^2 1; \qquad U - \varepsilon(U)1\,.
\end{aligned}$$

To prove this, we first verify by direct computations that the functionals X_{ij} and X_n annihilate all these elements. We omit the (boring) details of these computations. Thus $\mathcal{R}$ is contained in the right ideal of $\ker\varepsilon$ associated with (Γ, d). Let $\mathcal{E}$ be the eight-dimensional vector space spanned by the elements u^i_j for $i \neq j$, $u^1_1 - 1$ and $u^2_2 - 1$. From the above list of generators of $\mathcal{R}$ we conclude that each quadratic term $u^r_s u^i_j - \delta_{rs}\delta_{ij}1$ belongs to $\mathcal{E} + \mathcal{R}$. (For two missing terms $u^1_2u^2_3$ and $u^3_2u^2_1$ we get $u^1_2u^2_3-(q-\alpha)u^1_3\in\mathcal{R}$ and $u^3_2u^2_1-(q^{-1}-\beta)u^3_1\in\mathcal{R}$.) Hence we have codim $\mathcal{R}\le 8$. Since $\dim(\Gamma,\mathrm{d})=8$, $\mathcal{R}$ is the right ideal of $\ker\varepsilon$ associated with the calculus (Γ,d). Now we turn to the higher-order calculus of (Γ,d). Our first aim is to prove the following:

Lemma 4. *If $(\alpha,\beta)\ne(q,q^{-1})$ and $(\alpha,\beta)\ne(q^{-1},q)$, then we have $\omega_{13}\otimes\omega_{31}\in\mathcal{S}(\mathcal{R})$.*

Proof. Recall that the elements $u^1_2u^2_1$, $u^2_3u^3_2$ and $u^1_3u^3_1$ belong to the right ideal $\mathcal{R}$. We determine the corresponding symmetric elements. All functionals $X_{ij}X_{rs}$ with $i<j, r<s$ or $i>j, r>s$ and all functionals X_nX_{rs}, $X_{rs}X_n$, X_nX_m annihilate these elements. We have

$$\begin{aligned}\mathcal{S}(u^1_2u^2_1)&=X_{12}(u^1_2u^2_2)X_{21}(u^2_2u^2_1)\omega_{12}\otimes\omega_{21}+X_{13}(u^1_3u^2_2)X_{31}(u^3_2u^2_1)\omega_{13}\otimes\omega_{31}\\&\quad+X_{13}(u^1_2u^2_3)X_{31}(u^2_1u^3_2)\omega_{13}\otimes\omega_{31}+X_{21}(u^1_1u^2_1)X_{12}(u^1_2u^1_1)\omega_{21}\otimes\omega_{12}\\&=q\omega_{12}\otimes\omega_{21}+(q^{-1}-\beta)\omega_{13}\otimes\omega_{31}+(q-\alpha)\omega_{13}\otimes\omega_{31}\\&\quad+q^{-1}\omega_{21}\otimes\omega_{12}\\&=q\omega_{12}\otimes\omega_{21}+q^{-1}\omega_{21}\otimes\omega_{12}+(\lambda_+-\alpha-\beta)\omega_{13}\otimes\omega_{31}.\end{aligned}$$

Similarly, we obtain

$$\mathcal{S}(u^2_3u^3_2)=q\omega_{23}\otimes\omega_{32}+q^{-1}\omega_{32}\otimes\omega\omega_{23}+(\lambda_+-\alpha-\beta)\omega_{31}\otimes\omega_{13}$$

and

$$\mathcal{S}(u^1_3u^3_1)=q\omega_{13}\otimes\omega_{31}+q^{-1}\omega_{31}\otimes\omega_{13}.$$

Using two of these formulas and the facts that $\mathcal{S}(xy)=P_{\mathrm{inv}}(\mathcal{S}(x)y)$, $x\in\mathcal{R}$, by Lemma 1, and $P_{\mathrm{inv}}(y\eta)=\varepsilon(y)P_{\mathrm{inv}}(\eta)$ for $\eta\in\Gamma\otimes_{\mathcal{A}}\Gamma$ and $y\in\mathcal{A}$, we compute

$$\begin{aligned}\mathcal{S}(u^1_2u^2_1u^3_3)&=P_{\mathrm{inv}}(\mathcal{S}(u^1_2u^2_1)u^3_3)\\&=P_{\mathrm{inv}}(q\omega_{12}\otimes\omega_{21}u^3_3+q^{-1}\omega_{21}\otimes\omega_{12}u^3_3+(\lambda_+-\alpha-\beta)\omega_{13}\otimes\omega_{31}u^3_3)\\&=P_{\mathrm{inv}}(qu^3_3\omega_{12}\otimes\omega_{21}+q(q-\alpha)u^3_2\omega_{12}\otimes\omega_{21}+q^{-1}u^3_3\omega_{21}\otimes\omega_{12}\\&\qquad+q^{-1}(q-\alpha)u^3_2\omega_{21}\otimes\omega_{13}+q^{-1}(q-\alpha)(q-\beta)u^3_3\omega_{31}\otimes\omega_{13}\\&\qquad+(\lambda_+-\alpha-\beta)q^2u^3_3\omega_{13}\otimes\omega_{31})\\&=q\omega_{12}\otimes\omega_{21}+q^{-1}\omega_{21}\otimes\omega_{12}+q^{-1}(q-\alpha)(q-\beta)\omega_{31}\otimes\omega_{13}\\&\quad+q^2(\lambda_+-\alpha-\beta)\omega_{13}\otimes\omega_{31}\\&=\mathcal{S}(u^1_2u^2_1)+(q-\alpha)(q-\beta)\mathcal{S}(u^1_3u^3_1)+(1-q\alpha)(\beta-q^{-1})\omega_{13}\otimes\omega_{31},\end{aligned}$$

so that $\omega_{13}\otimes\omega_{31}\in\mathcal{S}(\mathcal{R})$ provided that $\alpha\ne q^{-1}$ and $\beta\ne q^{-1}$. Similarly, we get

$$\begin{aligned}\mathcal{S}(u^2_3u^3_2u^1_1)&=\mathcal{S}(u^2_3u^3_2)+(q^{-1}-\alpha)(q^{-1}-\beta)\mathcal{S}(u^1_3u^3_1)\\&\quad+(\alpha-q)(\beta-q)\omega_{13}\otimes\omega_{31},\end{aligned}$$

hence $\omega_{13}\otimes\omega_{31}\in\mathcal{S}(\mathcal{R})$ when $\alpha\ne q$ and $\beta\ne q$. □

Therefore, if $(\alpha, \beta) \neq (q^{-1}, q)$ and $(\alpha, \beta) \neq (q, q^{-1})$, then the 2-form $\omega_{13} \wedge \omega_{31}$ vanishes in $\Gamma_2^{\wedge}$ and hence the space of 2-forms does *not* give the corresponding space for the ordinary differential calculus on $SL(3)$ when $q \to 1$. The two remaining distinguished cases $(\alpha, \beta) = (q^{-1}, q)$ and $(\alpha, \beta) = (q, q^{-1})$ will be treated in Section 4. For this let (Γ_1, d) and (Γ_2, d) denote the first-order differential calculus (Γ, d) on $SL_q(3)$ with $(\alpha, \beta) = (q^{-1}, q)$ and $(\alpha, \beta) = (q, q^{-1})$, respectively.

4. The differential calculi (Γ_1, d) and (Γ_2, d) on $SL_q(3)$

Let $\mathcal{R}_r$ be the right ideal of $\ker \varepsilon$ and let $\mathcal{X}_r$ be the linear span of linear functionals $X_{ij}, i \neq j, i, j = 1, 2, 3$, and $X_n, n = 1, 2$, for the calculus $(\Gamma_r, \mathrm{d}), r = 1, 2$. From the definitions of functionals $X_{i,j}$, X_n and the commutation rules in the algebra $\mathcal{U}_q(sl_3)$ we obtain the following commutation relations for the generators of the quantum Lie algebra $\mathcal{X}_r$.

$$
\begin{aligned}
\mathcal{X}_1 \text{ and } \mathcal{X}_2:\quad & X_{12}X_{32} - q^{-1}X_{32}X_{12} = 0, \quad X_{23}X_{21} - q^{-1}X_{21}X_{23} = 0,\\
& X_{12}X_{21} - q^2X_{21}X_{12} = X_1,\\
& X_{13}X_{31} - q^2X_{31}X_{13} + q^{-1}\lambda X_1X_2 = X_1 + X_2,\\
& X_{23}X_{32} - q^2X_{32}X_{23} = X_2, \quad X_1X_2 - X_2X_1 = 0,\\
& X_1X_{12} - q^{-4}X_{12}X_1 = (1 + q^{-2})X_{12},\\
& X_1X_{21} - q^4X_{21}X_1 = -(q^2 + q^4)X_{21},\\
& X_2X_{23} - q^{-4}X_{23}X_2 = (1 + q^{-2})X_{23},\\
& X_2X_{32} - q^4X_{32}X_2 = -(q^2 + q^4)\, X_{32},\\
& X_1X_{23} - q^2X_{23}X_1 = -q^2X_{23}, \quad X_1X_{32} - q^{-2}X_{32}X_1 = X_{32},\\
& X_1X_{13} - q^{-2}X_{13}X_1 = X_{13}, \quad X_1X_{31} - q^2X_{31}X_1 = -q^2X_{31},\\
& X_2X_{13} - q^{-2}X_{13}X_2 = X_{13}, \quad X_2X_{31} - q^2X_{31}X_2 = -q^2X_{31},\\
& X_2X_{12} - q^2X_{12}X_2 = -q^2X_{12}, \quad X_2X_{21} - q^{-2}X_{21}X_2 = X_{21},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X}_1:\quad & X_{13}X_{23} - qX_{23}X_{13} = 0, \quad X_{32}X_{31} - q^{-1}X_{31}X_{32} = 0,\\
& X_{12}X_{13} - qX_{13}X_{12} = 0, \quad X_{31}X_{21} - q^{-1}X_{21}X_{31} = 0,\\
& X_{12}X_{23} - q^{-1}X_{23}X_{12} = X_{13}, \quad X_{32}X_{21} - qX_{21}X_{32} = X_{31},\\
& X_{12}X_{31} - qX_{31}X_{12} = -qX_{32},\\
& X_{13}X_{21} - qX_{21}X_{13} - \lambda X_{23}X_1 = -qX_{23},\\
& X_{13}X_{32} - qX_{32}X_{13} = X_{12},\\
& X_{23}X_{31} - qX_{31}X_{23} + q^{-1}\lambda X_{21}X_2 = X_{21}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X}_2:\quad & X_{13}X_{23} - q^{-1}X_{23}X_{13} = 0, \quad X_{32}X_{31} - qX_{31}X_{32} = 0,\\
& X_{12}X_{13} - q^{-1}X_{13}X_{12} = 0, \quad X_{31}X_{21} - qX_{21}X_{31} = 0,\\
& X_{12}X_{23} - qX_{23}X_{12} = X_{13}, \quad X_{32}X_{21} - q^{-1}X_{21}X_{32} = X_{31},\\
& X_{12}X_{31} - qX_{31}X_{12} - \lambda X_1X_{32} = -qX_{32},\\
& X_{13}X_{21} - qX_{21}X_{13} = -qX_{23},\\
& X_{13}X_{32} - qX_{32}X_{13} + q^{-1}\lambda X_2X_{12} = X_{12},\\
& X_{23}X_{31} - qX_{31}X_{23} = X_{21}.
\end{aligned}
$$

Next we shall describe the corresponding higher-order differential calculi. For this we need to know the vector space $\mathcal{S}(\mathcal{R}_r)$. Let I denote the ordered index set $\{1, 2, 21, 31, 32, 12, 13, 23\}$. The right ideal $\mathcal{R}_r$, $r = 1, 2$, has 37 generators which have been listed in the preceding section (recall that $\alpha = q^{-1}, \beta = q$ for $\mathcal{R}_1$ and $\alpha = q, \beta = q^{-1}$ for $\mathcal{R}_2$). Let $\mathcal{B}_r$ be the linear span of these generators. Using the formulas for the comultiplications of X_{ij} and X_n and for the pairing between $\mathcal{U}_q(sl_3)$ and $SL_q(3)$ one can compute the symmetric elements $\mathcal{S}(x)$ for the generators of $\mathcal{R}_r$. We state only the result of these (long) computations. The following 36 elements of $\Gamma \otimes_{\mathcal{A}} \Gamma$ belong to $\mathcal{S}(\mathcal{R}_r)$ and form a basis of the vector space $\mathcal{S}(\mathcal{B}_r)$.

$\mathcal{S}(\mathcal{R}_1)$ and $\mathcal{S}(\mathcal{R}_2)$: $\omega_{ij} \otimes \omega_{ij}$ for $i \neq j, i, j = 1, 2, 3$; $\omega_n \otimes \omega_n$ for $n = 1, 2$;

$$\omega_{12} \otimes \omega_{32} + q\omega_{32} \otimes \omega_{12},\ \omega_{23} \otimes \omega_{21} + q\omega_{21} \otimes \omega_{23},$$
$$\omega_{12} \otimes \omega_{21} + q^{-2}\omega_{21} \otimes \omega_{12},\ \omega_{13} \otimes \omega_{31} + q^{-2}\omega_{31} \otimes \omega_{13},$$
$$\omega_{23} \otimes \omega_{32} + q^{-2}\omega_{32} \otimes \omega_{23},$$
$$\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1 - q^{-1}\lambda\omega_{13} \otimes \omega_{31},$$
$$\omega_1 \otimes \omega_{12} + q^4\omega_{12} \otimes \omega_1,\quad \omega_1 \otimes \omega_{21} + q^{-4}\omega_{21} \otimes \omega_1,$$
$$\omega_2 \otimes \omega_{23} + q^4\omega_{23} \otimes \omega_2,\quad \omega_2 \otimes \omega_{32} + q^{-4}\omega_{32} \otimes \omega_2,$$
$$\omega_1 \otimes \omega_{23} + q^{-2}\omega_{23} \otimes \omega_1,\quad \omega_1 \otimes \omega_{32} + q^2\omega_{32} \otimes \omega_1,$$
$$\omega_1 \otimes \omega_{13} + q^2\omega_{13} \otimes \omega_1,\quad \omega_1 \otimes \omega_{31} + q^{-2}\omega_{31} \otimes \omega_1,$$
$$\omega_2 \otimes \omega_{13} + q^2\omega_{13} \otimes \omega_2,\quad \omega_2 \otimes \omega_{31} + q^{-2}\omega_{31} \otimes \omega_2,$$
$$\omega_2 \otimes \omega_{12} + q^{-2}\omega_{12} \otimes \omega_2,\quad \omega_2 \otimes \omega_{21} + q^2\omega_{21} \otimes \omega_2,$$

$\mathcal{S}(\mathcal{R}_1)$:
$$\omega_{13} \otimes \omega_{23} + q^{-1}\omega_{23} \otimes \omega_{13},\quad \omega_{32} \otimes \omega_{31} + q\omega_{31} \otimes \omega_{32},$$
$$\omega_{12} \otimes \omega_{13} + q^{-1}\omega_{13} \otimes \omega_{12},\quad \omega_{31} \otimes \omega_{21} + q\omega_{21} \otimes \omega_{31},$$
$$\omega_{12} \otimes \omega_{23} + q\omega_{23} \otimes \omega_{12},\quad \omega_{32} \otimes \omega_{21} + q^{-1}\omega_{21} \otimes \omega_{32},$$
$$\omega_{12} \otimes \omega_{31} + q^{-1}\omega_{31} \otimes \omega_{12},$$
$$\omega_{23} \otimes \omega_1 + q^2\omega_1 \otimes \omega_{23} + \lambda\omega_{13} \otimes \omega_{21},$$
$$\omega_{13} \otimes \omega_{32} + q^{-1}\omega_{32} \otimes \omega_{13},$$
$$\omega_2 \otimes \omega_{21} + q^2\omega_{21} \otimes \omega_2 + \lambda\omega_{31} \otimes \omega_{23}.$$

$\mathcal{S}(\mathcal{R}_2)$:
$$\omega_{13} \otimes \omega_{23} + q\omega_{23} \otimes \omega_{13},\quad \omega_{32} \otimes \omega_{31} + q^{-1}\omega_{31} \otimes \omega_{32},$$
$$\omega_{12} \otimes \omega_{13} + q\omega_{13} \otimes \omega_{12},\quad \omega_{31} \otimes \omega_{21} + q^{-1}\omega_{21} \otimes \omega_{31},$$
$$\omega_{12} \otimes \omega_{23} + q^{-1}\omega_{23} \otimes \omega_{12},\quad \omega_{32} \otimes \omega_{21} + q\omega_{21} \otimes \omega_{32},$$
$$\omega_1 \otimes \omega_{32} + q^2\omega_{32} \otimes \omega_1 + \lambda\omega_{12} \otimes \omega_{31},$$
$$\omega_{13} \otimes \omega_{12} + q^{-1}\omega_{21} \otimes \omega_{21},$$
$$\omega_{12} \otimes \omega_2 + q^2\omega_2 \otimes \omega_{12} + \lambda\omega_{32} \otimes \omega_{13},$$
$$\omega_{23} \otimes \omega_{31} + q^{-1}\omega_{31} \otimes \omega_{23}.$$

The preceding formulas showed that both FODC (Γ_1, d) and (Γ_2, d) are very close to the classical differential calculus on $SL(3)$. For instance, except for two cases (that is, $\omega_{12}u^n_3, \omega_{32}u^n_1$ for $r = 1$ and $\omega_{23}u^n_2, \omega_{21}u^n_2$ for $r = 2$), each 1-form $\omega_i u^n_m$ is equal to $u^n_m\omega_i$ multiplied by some power of q. Except for three cases, the commutation relations of the quantum Lie algebra $\mathcal{X}_r$ and the elements of the left module basis of S_2 contain only two quadratic terms as in the classical case.

Lemma 5. *For $r = 1, 2$, we have $\mathcal{S}(\mathcal{B}_r) = \mathcal{S}(\mathcal{R}_r)$.*

Proof. By Lemma 2 (ii), it suffices to show that $P_{\mathrm{inv}}(\zeta u^i_j) \in \mathcal{S}(\mathcal{B}_r)$ for $i, j = 1, 2, 3$ and for all 36 basis elements ζ of listed above. This is obviously fulfilled if ζu^i_j is a scalar multiple of $u^i_j \zeta$, since $P_{\mathrm{inv}}(a\zeta) = \varepsilon(a) P_{\mathrm{inv}}(\zeta)$ by Lemma 2.2 in [18]. From the commutation relations between matrix entries and 1-forms we conclude that it remains to check the condition $P_{\mathrm{inv}}(\zeta u^i_j) \in \mathcal{S}(\mathcal{B}_r)$ for all basis elements ζ of $\mathcal{S}(\mathcal{B}_r)$ which are sums of three terms or contain the 1-forms ω_{12} or ω_{32} in case $r = 1$ resp. ω_{23} or ω_{21} in case $r = 2$. These are 14 elements ζ in either case $r = 1$ and $r = 2$. We carry out this verification for the two elements $\zeta_1 := \omega_{12} \otimes \omega_{32} + q\omega_{32} \otimes \omega_{12}$ and $\zeta_2 := \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1 - q^{-1}\lambda\omega_{13} \otimes \omega_{31}$ in case $r = 1$. From $\omega_{12} \otimes \omega_{32} u^i_3 = q u^i_3 \omega_{12} \otimes \omega_{32} + q\lambda u^i_2 \omega_{13} \otimes \omega_{32}$ and $\omega_{32} \otimes \omega_{12} u^i_3 = q u^i_3 \omega_{32} \otimes \omega_{12} + q^{-1}\lambda u^i_2 \omega_{32} \otimes \omega_{13}$ it follows that $P_{\mathrm{inv}}(\zeta_1 u^3_3) = q\zeta_1$ and $P_{\mathrm{inv}}(\zeta_1 u^2_3) = q\lambda(\omega_{13} \otimes \omega_{32} + q^{-1}\omega_{32} \otimes \omega_{13}) \in \mathcal{S}(B_r)$. Moreover, $\zeta_1 u^i_j$ is a multiple of $u^i_j \zeta_1$ for all elements u^i_j other than u^3_3 and u^2_3. For ζ_2 we obtain $\zeta_2 u^i_j = q^{-2\delta_{j1} + 2\delta_{j3}} u^i_j \zeta_2$, so the condition is also valid for ζ_2. The other 26 cases are treated in a similar way. □

Thus, by Lemma 5, the 36 basis elements for the vector space $\mathcal{S}(\mathcal{B}_r)$ of the above list form a free left module basis for $S_2 = \mathcal{A}\mathcal{S}(\mathcal{R}_r)\mathcal{A}$. This gives a precise description of the $\mathcal{A}$ sub-bimodule S_2 of $\Gamma \otimes_{\mathcal{A}} \Gamma$. From the definitions of functionals X_{ij}, X_n and the pairing between $\mathcal{U}_q(sl_3)$ and $SL_q(3)$ it follows that for both calculi

$$\omega(u^1_1) = \omega_1, \qquad \omega(u^2_2) = \omega_2 - q^2\omega_1,$$
$$\omega(u^3_3) = -q^2\omega_2 \quad \text{and} \quad \omega(u^i_j) = \omega_{ij} \quad \text{for } i \neq j.$$

Recall that for any left-covariant differential calculus over $\mathcal{A}$ we have $\mathrm{d}\omega(a) = -\omega(a_{(1)}) \wedge \omega(a_{(2)})$, $a \in \mathcal{A}$. From the preceding we obtain the following *Maurer–Cartan formulas* for our differential calculi:

(Γ_1, d) and (Γ_2, d):
$$\begin{aligned}
\mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_{21} - \omega_{13} \wedge \omega_{31},\\
\mathrm{d}\omega_2 &= -\omega_{13} \wedge \omega_{31} - \omega_{23} \wedge \omega_{32},\\
\mathrm{d}\omega_{13} &= -\omega_1 \wedge \omega_{13} - \omega_2 \wedge \omega_{13} - \omega_{12} \wedge \omega_{23},\\
\mathrm{d}\omega_{23} &= q^2\omega_1 \wedge \omega_{23} - (1 + q^{-2})\omega_2 \wedge \omega_{23} + q\omega_{13} \wedge \omega_{21},\\
\mathrm{d}\omega_{21} &= (q^2 + q^4)\omega_1 \wedge \omega_{21} - \omega_2 \wedge \omega_{21} - \omega_{23} \wedge \omega_{31}.
\end{aligned}$$

(Γ_1, d):
$$\begin{aligned}
\mathrm{d}\omega_{12} &= -(1 + q^{-2})\omega_1 \wedge \omega_{12} + q^2\omega_2 \wedge \omega_{12} - \omega_{13} \wedge \omega_{32},\\
\mathrm{d}\omega_{31} &= q^2\omega_1 \wedge \omega_{31} + q^2\omega_2 \wedge \omega_{31} + q^{-1}\omega_{21} \wedge \omega_{32},\\
\mathrm{d}\omega_{32} &= -\omega_1 \wedge \omega_{32} + (q^2 + q^4)\omega_2 \wedge \omega_{32} + q\omega_{12} \wedge \omega_{31}.
\end{aligned}$$

(Γ_2, d):
$$\begin{aligned}
\mathrm{d}\omega_{12} &= -(1 + q^{-2})\omega_1 \wedge \omega_{12} + q^2\omega_2 \wedge \omega_{12} - q^2\omega_{13} \wedge \omega_{32},\\
\mathrm{d}\omega_{31} &= q^2\omega_1 \wedge \omega_{31} + q^2\omega_2 \wedge \omega_{31} + q\omega_{21} \wedge \omega_{32},\\
\mathrm{d}\omega_{32} &= -\omega_1 \wedge \omega_{32} + (q^2 + q^4)\omega_2 \wedge \omega_{32} + q^{-1}\omega_{12} \wedge \omega_{31}.
\end{aligned}$$

In this and the next paragraph we omit the subindex r which refers to one of the calculi (Γ_1, d) and (Γ_2, d). We define a 64×64 matrix $\sigma = (\sigma^{\mathrm{ij}}_{\mathrm{mn}})_{\mathrm{i,j,n,m} \in I}$ as follows. Consider elements $\omega_{\mathrm{n}} \otimes \omega_{\mathrm{m}} + \mu\omega_{\mathrm{m}} \otimes \omega_{\mathrm{n}}$ and $\omega_{\mathrm{i}} \otimes \omega_{\mathrm{j}} + \gamma\omega_{\mathrm{j}} \otimes \omega_{\mathrm{i}} + \delta\omega_{\mathrm{n}} \otimes \omega_{\mathrm{m}}$ (written in the order of the index set I, i.e. $\mathrm{n} < \mathrm{m}$ and $\mathrm{i} < \mathrm{j}$) of our vector space basis of $\mathcal{S}(\mathcal{R})$. We set $\sigma^{\mathrm{mn}}_{\mathrm{nm}} = \mu$, $\sigma^{\mathrm{nm}}_{mn} = \mu^{-1}$, $\sigma^{\mathrm{ji}}_{\mathrm{ij}} = \gamma$, $\sigma^{\mathrm{ij}}_{\mathrm{ji}} = \gamma^{-1}$, $\sigma^{\mathrm{nm}}_{\mathrm{ij}} = \delta$ and $\sigma^{\mathrm{mn}}_{\mathrm{ji}} = -\delta\mu\gamma^{-1}$. The number $\sigma^{\mathrm{ii}}_{\mathrm{ii}}$, $\mathrm{i} \in I$, are set equal to 1 and the remaining matrix entries are set zero. Then the 36 basis elements of $\mathcal{S}(\mathcal{R})$ are $\omega_{\mathrm{i}} \otimes \omega_{\mathrm{j}} + \sum_{\mathrm{n,m}} \sigma^{\mathrm{nm}}_{\mathrm{ij}} \omega_{\mathrm{n}} \otimes \omega_{\mathrm{m}}$, $\mathrm{i}, \mathrm{j} \in I$, $\mathrm{i} \leq \mathrm{j}$. From the above formulas we see that the commutation relations of the quantum Lie algebra generators can be expressed as $X_{\mathrm{i}} X_{\mathrm{j}} - \sum_{\mathrm{n,n}} \sigma^{\mathrm{ij}}_{\mathrm{nm}} X_{\mathrm{n}} X_{\mathrm{m}} = \sum_{\mathrm{t}} C^{\mathrm{t}}_{\mathrm{ij}} X_{\mathrm{t}}$, $\mathrm{i} \neq \mathrm{j}$, with certain coefficients $C^{\mathrm{t}}_{\mathrm{ij}} \in \mathbb{C}$. The linear transformation σ of the 64-dimensional vector space $\Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}}$ has eigenvalues 1 and -1 with multiplicities 36 and 28, respectively. Obvoiusly, σ^2 is the identity. However, σ does *not* satisfy the braid relation $\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$ on $\Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}} \otimes \Gamma_{\mathrm{inv}}$. (For instance, we have $\sigma_{12}\sigma_{23}\sigma_{12}(\omega_1 \otimes \omega_{23} \otimes \omega_{13}) = q\omega_{13} \otimes \omega_{23} \otimes \omega_1 - q^{-2}\lambda\omega_{13} \otimes \omega_{21} \otimes \omega_{13}$ and $\sigma_{23}\sigma_{12}\sigma_{23}(\omega_1 \otimes \omega_{23} \otimes \omega_{13}) = q\omega_{13} \otimes \omega_{23} \otimes \omega_1 - \lambda\omega_{13} \otimes \omega_{21} \otimes \omega_{13}$ for the calculus (Γ_1, d).) Let $J(\mathcal{R}) = \oplus_n J_n(\mathcal{R})$ be the two-sided ideal of the tensor algebra $\Gamma^{\otimes}_{\mathrm{inv}}$ of the vector space Γ_{inv} which is generated by the set $\mathcal{S}(\mathcal{R})$. Clearly, $(\Gamma^{\wedge})_{\mathrm{inv}}$ is (isomorphic to) the quotient algebra $\Gamma^{\otimes}_{\mathrm{inv}}/J(\mathcal{R}) = \oplus_n \Gamma^{\otimes n}_{\mathrm{inv}}/J_n(\mathcal{R})$. The above basis elements of the vector space $\mathcal{S}(\mathcal{R})$ form a Gröbner basis (see, e.g. [6] for this concept) of the ideal $J(\mathcal{R})$ with respect to the above ordering of the index set I. We have checked this by using the computer algebra system FELIX [1]. (One may also verify this assertion by performing explicit calculations.) Therefore, we get a vector space basis for the quotient space $\Gamma^{\oplus}_{\mathrm{inv}}/J(\mathcal{R})$ by taking all monomials in the 1-forms ω_{i}, $\mathrm{i} \in I$, which are not multiples of the leading term of one of these Gröbner basis elements. From the special form of these elements we see that this set of monomials is the same as in the case $q = 1$. Hence the dimension of the vector space $\Gamma^{\otimes n}_{\mathrm{inv}}/J_n(\mathcal{R})$ is equal to the corresponding dimension $\binom{8}{n}$ in the classical case. Moreover, it follows that the associated higher-order calculi of both calculi (Γ_1, d) and (Γ_2, d) give the ordinary higher-order calculus on $SL(3)$ in the limit $q \to 1$.

5. Left-covariant differential calculi on $SL_q(N)$

Let $L^+ = ({}^+l^i_j)$ and $L^- = ({}^-l^i_j)$ be the $N \times N$ matrices of linear functionals ${}^+l^i_j$ and ${}^-l^i_j$ on $\mathcal{A} := SL_q(N)$ as defined in [5]. Recall that $L^\pm$ is uniquely determined by the properties that $L^\pm : \mathcal{A} \to M_N(\mathbb{C})$ is a unital algebra homomorphism and ${}^\pm l^i_j(u^n_m) = p^{\mp 1}(\hat{R}^{\pm 1})^{in}_{mj}$ for $i, j, n, m = 1, \ldots, N$, where p is an Nth root of q and $\hat{R}$ is the R-matrix of the quantum group $SL_q(N)$. In this section we define $(N^2 - 1)$-dimensional left-covariant FODC (Γ_1, d) and (Γ_2, d) over $\mathcal{A} = SL_q(N)$. They generalize the two calculi over $SL_q(N)$ studied in the preceding section. For this we set

$$X_{ij} = \lambda^{-1}\kappa({}^-l^j_i){}^-l^i_i \quad \text{and} \quad X_{ji} = -\lambda^{-1}\kappa({}^+l^i_j){}^+l^j_j \qquad \text{for } i < j \text{ and } r = 1,$$

$$X_{ij} := -\lambda^{-1+}l^j_j{}^-l^j_i \quad \text{and} \quad X_{ji} = \lambda^{-1-}l^i_i{}^+l^i_j \qquad \text{for } i < j \text{ and } r = 2,$$

$$X_n = q\lambda^{-1}(\varepsilon - ({}^-l^n_n{}^+l^{n+1}_{n+1})^2) \quad \text{for } n = 1, \ldots, N-1 \text{ and } r = 1, 2.$$

Let $\mathcal{X}_r$ denote the linear span of functionals $X_{ij}, i \neq j,\ i, j = 1, \ldots, N$, and $X_n, n = 1, \ldots, N-1$. Computing the comultiplications of these generators, we obtain:

for $r = 1$ and $i < j$:

$$\Delta X_{ij} = \varepsilon \otimes X_{ij} + \sum_{m=i+1}^{j} X_{im} \otimes {}^{-}l^i_i \kappa({}^{-}l^j_m)$$

and

$$\Delta X_{ji} = \varepsilon \otimes X_{ji} + \sum_{m=i}^{j-1} X_{jm} \otimes {}^{+}l^j_j \kappa({}^{+}l^i_m);$$

for $r = 2$ and $i < j$:

$$\Delta X_{ij} = \varepsilon \otimes X_{ij} + \sum_{m=i}^{j-1} X_{mj} \otimes {}^{+}l^j_j {}^{-}l^m_i$$

and

$$\Delta X_{ji} = \varepsilon \otimes X_{ji} + \sum_{m=i+1}^{j} X_{mi} \otimes {}^{-}l^i_i {}^{+}l^m_j;$$

for $r = 1, 2$:

$$\Delta X_n = \varepsilon \otimes X_n + X_n \otimes ({}^{-}l^n_n {}^{+}l^{n+1}_{n+1})^2 .$$

In particular, these formulas show that $\Delta X - \varepsilon \otimes X \in \mathcal{X}_r \otimes \mathcal{A}'$ for all $X \in \mathcal{X}_r$ and $r = 1, 2$. Therefore, by Lemma 1, the vector space $\mathcal{X}_r$ defines a left-covariant FODC over $SL_q(N)$. Furthermore, we verify that $X_{ij}(u^r_s) = \delta_{ir}\delta_{js}$ for $i \neq j$ and $X_n(u^r_s) = \delta_{rs}(\delta_{nr} - q^2\delta_{n+1,r})$ for $n = 1, \ldots, N-1$. Thus all elements of the quantum Lie algebra $\mathcal{X}_r$ annihilate the quantum trace $U = \sum q^{-2i} u^i_i$. Since $\dim \mathcal{X}_r = N^2 - 1$, the FODC (Γ_r, d) has dimension $N^2 - 1$.

It is not difficult to check that in the classical limit $q \to 1$ both FODC (Γ_r, d) give the ordinary FODC on $SL(N)$. (As in the preceding sections, we define the limit $q \to 1$ of the calculus (Γ_r, d) by keeping the Maurer–Cartan basis $\omega(u^r_s)$, $(r, s) \neq (N, N)$, of $(\Gamma_r)_{\mathrm{inv}}$ fixed.) Some computations show that for $|i - j| \geq 3$ and $r = 1, 2$ the 2-form $\omega_{ij} \wedge \omega_{ji}$ vanishes in $(\Gamma_r)^\wedge_2$. That is, if $N \geq 4$, both associated higher-order calculi do *not* have the classical higher-order calculus on $SL(N)$ as their limits when $q \to 1$. To overcome this disadvantage, we have taken up another approach in [14].

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